Corrections Exercices Séance 6

exercice 1

1/ Première solution:

- on considère $t < -1/2$,

$\mathbb{1}_{[-1/2, 1/2[}(t) = 0$

$H(t+1/2) = 0$

$H(t-1/2) = 0$

donc on a bien $\mathbb{1}_{[-1/2, 1/2]}(t) = H(t+1/2) - H(t-1/2)$

- si $t \in [-1/2, 1/2[$

$\mathbb{1}_{[-1/2, 1/2[}(t) = 1$

$H(t+1/2) = 1$

$H(t-1/2) = 0$

- Si $t \geq 1/2$,

$\mathbb{1}_{[-1/2, 1/2[}(t) = 0$

$H(t+1/2) = 1$

$H(t-1/2) = 1$

Finalement pour $t \in \mathbb{R}$, $\mathbb{1}_{[-1/2, 1/2]}(t) = H(t+1/2) - H(t-1/2)$.

Deuxieme solution

$[-1/2, +\infty[= [-1/2, 1/2[\cup]1/2, +\infty[$.

et $[-1/2, 1/2] \cap]1/2, +\infty[= \emptyset$

donc $\mathbb{1}_{[-1/2, +\infty[}(t) = \mathbb{1}_{[-1/2, 1/2[}(t) + \mathbb{1}_{[1/2, +\infty[}(t)$

D'où $\mathbb{1}_{[-1/2, 1/2[}(t) = \mathbb{1}_{[-1/2, +\infty[}(t) - \mathbb{1}_{[1/2, +\infty[}(t) = H(t+1/2) - H(t-1/2)$

2/ $TF[H(t+\tfrac{1}{2})] = e^{i\pi\nu t}\, TF[H(t)]$

$TF[H(t-\tfrac{1}{2})] = e^{-i\pi\nu t}\, TF[H(t)]$

donc $TF[1_{[-1/2,\,1/2]}(t)]$

$= TF[H(t)]\,(e^{i\pi\nu t} - e^{-i\pi\nu t})$

$= TF[H(t)]\ \sin \pi\nu t \times 2i$

3/ $\sin(\pi\nu)\,\delta(\nu)$ est une distribution.
Appliquée à n'importe quelle fonction

$$\int_{-\infty}^{+\infty} \sin(\pi\nu)\,\delta(\nu)\,F(\nu)\,d\nu = \sin(\pi 0)\,F(0) = 0$$

indépendamment de F

donc $\sin(\pi\nu)\,\delta(\nu) = 0$.

4/ Comme $\dfrac{\sin \pi\nu}{\pi\nu}$ est une fonction intégrable,

$$vp\left(\frac{1}{\nu}\right) \sin \pi\nu = \frac{\sin(\pi\nu)}{\nu}$$

D'où $\dfrac{1}{2i\pi}\, vp\left(\dfrac{1}{\nu}\right) \sin \pi\nu \times 2i = \dfrac{\sin \pi\nu \times 2i}{2i\pi\nu}$

$= \dfrac{\sin \pi\nu}{\pi\nu}$

exercice 2

1. On pose $\Pi(t) = \mathbb{1}_{[-1/2, 1/2]}(t)$

$$\Pi\left(\frac{t}{4}\right) = \mathbb{1}_{[-2, 2]}(t)$$

$$\Pi\left(\frac{t-1}{4}\right) = \mathbb{1}_{[-2, 2]}(t-1) = \mathbb{1}_{[-1, 3]}(t)$$

Donc $x_a(t) = 2\,\Pi\left(\frac{t-1}{4}\right)$

$$TF\left[\Pi(t)\right] = \frac{\sin \pi\nu}{\pi\nu}$$

$$TF\left[\Pi\left(\frac{t}{4}\right)\right] = 4.\frac{\sin \pi 4\nu}{\pi 4\nu} = \frac{\sin 4\pi\nu}{\pi\nu}$$

$$TF\left[\Pi\left(\frac{t-1}{4}\right)\right] = e^{-2i\pi\nu}\,\frac{\sin 4\pi\nu}{\pi\nu}$$

$$X_a(\nu) = 2e^{-2i\pi\nu}\,\frac{\sin 4\pi\nu}{\pi\nu}$$

2. $\mathbb{1}_{[0,1]}(t) = \Pi(t - 1/2)$

Donc $TF\left[\mathbb{1}_{[0,1]}(t)\right](\nu) = \frac{\sin \pi\nu}{\pi\nu}\,e^{-i\pi\nu}$

$$= \frac{\left(e^{i\pi\nu} - e^{-i\pi\nu}\right)e^{-i\pi\nu}}{2i\pi\nu}$$

$$= \frac{1 - e^{-2i\pi\nu}}{2i\pi\nu}$$

$$\frac{d}{d\nu}\left[\frac{1 - e^{-2i\pi\nu}}{2i\pi\nu}\right] = \frac{2i\pi e^{-2i\pi\nu}(2i\pi\nu) - (1 - e^{-2i\pi\nu})(2i\pi)}{-4\pi^2\nu^2}$$

$$\frac{d}{d\nu}\left[\frac{1 - e^{-2i\pi\nu}}{2i\pi\nu}\right] = \frac{2i\pi\nu e^{-2i\pi\nu} - 1 + e^{-2i\pi\nu}}{2i\pi\nu^2}$$

$$TF\left[(-2i\pi t)\, \mathbb{1}_{[0,1]}(t)\right] = \frac{d}{d\nu}\left(TF\left[\mathbb{1}_{[0,1]}(t)\right](\nu)\right)$$

Donc

$$X_b(\nu) = \frac{-1}{2i\pi} \times \frac{2i\pi\nu e^{-2i\pi\nu} - 1 + e^{-2i\pi\nu}}{2i\pi\nu^2}$$

$$X_b(\nu) = \frac{(1 + 2i\pi\nu)e^{-2i\pi\nu} - 1}{4\pi^2\nu^2}$$

3. * $TF[x(-t)](\nu) = \int_{-\infty}^{+\infty} x(-t)e^{-2i\pi\nu t}\,dt$

changement de variable $t' = -t$

$$TF[x(-t)](\nu) = \int_{-\infty}^{+\infty} x(t')e^{2\pi i(-\nu)t'}\,dt'$$

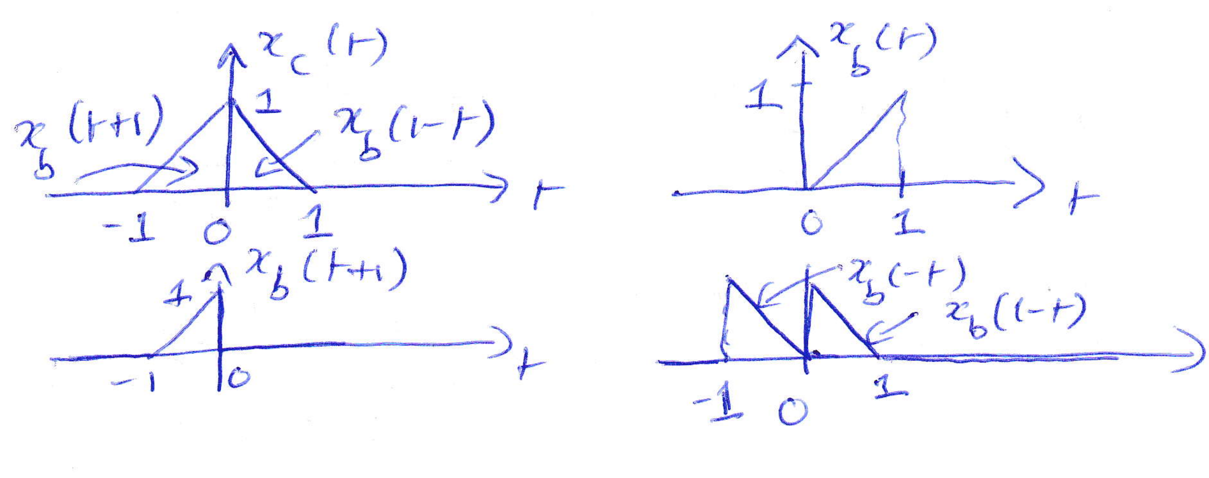
$$= TF[x(t)](-\nu)$$

* Je pose $x''(t) = x'(t-1) = x(1-t)$

$x'(t) = x(-t)$ donc $X'(\nu) = X(-\nu)$

$X''(\nu) = X'(\nu)e^{-2i\pi\nu} = X(-\nu)e^{-2i\pi\nu}$

4.



S_6, S_5

5. $TF[x_b(t+1)] = X_b(\nu)e^{2i\pi\nu}$

$TF[x_b(1-t)] = X_b(-\nu)e^{-2i\pi\nu}$

Donc $X_c(\nu) = X_b(\nu)e^{2i\pi\nu} + X_b(-\nu)e^{-2i\pi\nu}$

6. $X_b(\nu)e^{2i\pi\nu} = \dfrac{1+2i\pi\nu - e^{2i\pi\nu}}{4\pi^2\nu^2}$

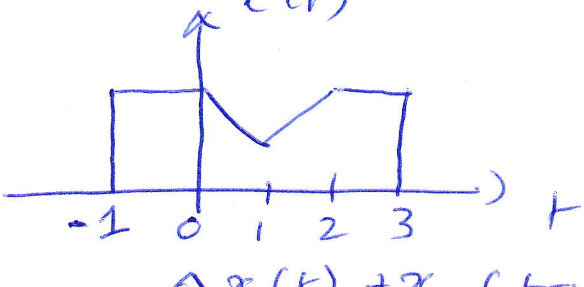
$X_b(-\nu)e^{-2i\pi\nu} = \dfrac{1-2i\pi\nu - e^{-2i\pi\nu}}{4\pi^2\nu^2}$

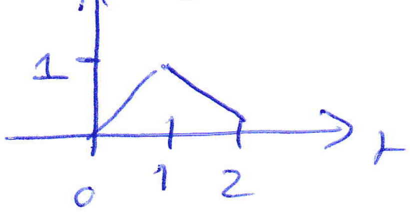
Donc $X_c(\nu) = \dfrac{2 - e^{2i\pi\nu} - e^{-2i\pi\nu}}{4\pi^2\nu^2}$

$X_c(\nu) = \dfrac{-(e^{i\pi\nu} - e^{-i\pi\nu})^2}{4\pi^2\nu^2}$

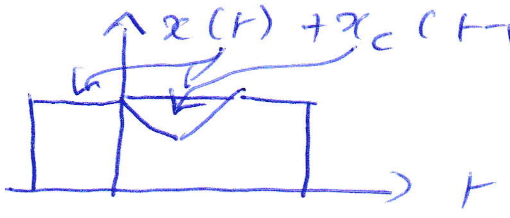
$X_c(\nu) = \dfrac{-(2i\sin\pi\nu)^2}{4\pi^2\nu^2} = \dfrac{\sin^2\pi\nu}{\pi^2\nu^2}$

7.





$x(t) + x_c(t-1) = x_a(t)$.



D'où $x(t) = x_a(t) - x_c(t-1)$

8. $TF[x_c(t-1)] = \dfrac{\sin^2 \pi\nu}{\pi^2\nu^2} \times e^{-2i\pi\nu}$

Donc $X(\nu) = \left(2\dfrac{\sin 4\pi\nu}{\pi\nu} - \dfrac{\sin^2 \pi\nu}{\pi^2\nu^2}\right) e^{-2i\pi\nu}$