exercice 1

1. $$S_N = \sum_{n=-N}^{N} c_n = c_0 + \sum_{n=1}^{N} (c_n + c_{-n})$$

$$S_N = (1-e^{-1})\left[1 + \sum_{n=1}^{N}\left(\frac{1}{1+2i\pi n} + \frac{1}{1-2i\pi n}\right)\right]$$

$$S_N = (1-e^{-1})\left[1 + \sum_{n=1}^{N} \frac{2}{1+4\pi^2 n^2}\right]$$

2. $\sum_{n=1}^{\infty} \frac{1}{1+4\pi^2 n^2}$ converge donc S_N converge.

$$\lim_{N\to+\infty} S_N = (1-e^{-1})\left(1 + 2\times\frac{1}{4}\left(\frac{3-e}{e-1}\right)\right)$$

$$\lim_{N\to+\infty} S_N = \frac{e-1}{e}\,\frac{\left(2(e-1)+3-e\right)}{2(e-1)} = \frac{1}{2e}(e+1)$$

$$\lim_{N\to+\infty} S_N = \frac{1}{2}(1+e^{-1})$$

3. Pour $t>0$, ~~et~~ $t<2$, $x(t) = e^{-\frac{t}{2}}$

$$\lim_{t\to 0^+} x(t) = 1.$$

Pour $t<0$ et $t>-2$, $x(t) = x(t+2) = e^{-\left(\frac{t+2}{2}\right)}$

$$\lim_{t\to 0^-} x(t) = e^{-\frac{2}{2}} = e^{-1}$$

Donc $\frac{1}{2}\lim_{t\to 0^+} x(t) + \frac{1}{2}\lim_{t\to 0^-} x(t) = \frac{1}{2}(1+e^{-1})$.

4. $x(t)$ est à variations bornées

donc $\sum_{n=-N}^{N} c_n e^{int} \to \frac{1}{2}\lim_{\tau\to t^+} x(\tau) + \frac{1}{2}\lim_{\tau\to t^-} x(\tau)$

quand $N\to+\infty$

On applique ce résultat à $t=0$.

exercice 2

$$X(\nu) = \int_{-\infty}^{+\infty} e^{-|t|} e^{-2i\pi\nu t} dt$$

$$X(\nu) = \int_{-\infty}^{0} e^{t} e^{-2i\pi\nu t} dt + \int_{0}^{+\infty} e^{-t} e^{-2i\pi\nu t} dt$$

$$X(\nu) = \left[\frac{e^{t-2i\pi\nu t}}{1-2i\pi\nu} \right]_{-\infty}^{0} + \left[\frac{e^{-t-2i\pi\nu t}}{-1-2i\pi\nu} \right]_{0}^{+\infty}$$

$$X(\nu) = \frac{1}{1-2i\pi\nu} + \frac{1}{1+2i\pi\nu}$$

$$X(\nu) = \frac{2}{1+4\pi^2\nu^2}.$$

exercice 3

1. $X(\nu) = \int_{-\infty}^{+\infty} x(t) e^{-2i\pi\nu t} dt = \int_{-\infty}^{+\infty} e^{-\pi t^2 - 2i\pi\nu t} dt$

Il se trouve que

$$t^2 + 2i\nu t = (t+i\nu)^2 + \nu^2$$

Donc $X(\nu) = \int_{-\infty}^{+\infty} e^{-\pi(t+i\nu)^2} e^{-\pi\nu^2} dt$

$$X(\nu) = e^{-\pi\nu^2} \int_{-\infty}^{+\infty} e^{-\pi(t+i\nu)^2} dt$$

2. La fonction $z \mapsto e^{-\pi z^2}$ est holomorphe comme composée de $z \mapsto z^2$ puis $z \mapsto e^{-\pi z}$ pour $z \in \mathbb{C}$

Donc $\int_{-\infty}^{+\infty} e^{-\pi(t+i\nu)^2} dt = \int_{-\infty}^{+\infty} e^{-\pi t^2} dt$.

3. $X(\nu) = e^{-\pi\nu^2} \int_{-\infty}^{+\infty} e^{-\pi t^2} dt$

$$X(\nu) = e^{-\pi\nu^2} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{\pi}}\right)^2}} dt$$

$$= e^{-\pi\nu^2} \times \sqrt{2\pi} \times \frac{1}{\sqrt{2}\sqrt{\pi}} = e^{-\pi\nu^2}$$

<u>exercice 4</u>

1. $|X(\nu)| = X(\nu) = e^{-\pi\nu^2}$

$\max_\nu X(\nu) = 1$

$$X(\nu) = \frac{1}{2}\max_\nu X(\nu) \iff e^{-\pi\nu^2} = \frac{1}{2} \iff -\pi\nu^2 = -\ln 2$$
$$\iff \nu^2 = \frac{1}{\pi}\ln 2$$
$$\iff \nu = \pm\sqrt{\frac{1}{\pi}\ln 2} \;;$$

$$\Delta\nu_x = \sqrt{\frac{\ln 2}{\pi}} - \left(-\sqrt{\frac{\ln 2}{\pi}}\right) = 2\sqrt{\frac{\ln 2}{\pi}}.$$

2. $|X(\nu)|^2 = X(\nu)^2 = e^{-2\pi\nu^2}$

$\max_\nu |X(\nu)|^2 = 1.$

$$|X(\nu)|^2 = \frac{1}{2}\max_\nu |X(\nu)|^2 \iff e^{-2\pi\nu^2} = \frac{1}{2}$$
$$\iff \nu^2 = \frac{\ln 2}{2\pi}$$
$$\iff \nu = \pm\sqrt{\frac{\ln 2}{2\pi}}.$$

$$\Delta\nu_0 = \sqrt{\frac{\ln 2}{2\pi}} - \left(-\sqrt{\frac{\ln 2}{2\pi}}\right) = 2\sqrt{\frac{\ln 2}{2\pi}} = \sqrt{\frac{2\ln 2}{\pi}}.$$

3. $\max_t |x(t)|^2 = \max_t e^{-\pi t^2} = 1$

$$|x(t)|^2 = \frac{1}{2} \iff e^{-\pi 2 t^2} = \frac{1}{2} \iff 2\pi t^2 = \ln 2$$
$$\iff t^2 = \frac{\ln 2}{2\pi} \iff t = \pm\sqrt{\frac{\ln 2}{2\pi}}$$

$$\Delta t_0 = \sqrt{\frac{\ln 2}{2\pi}} - \left(-\sqrt{\frac{\ln 2}{2\pi}}\right) = 2\sqrt{\frac{\ln 2}{2\pi}} = \sqrt{\frac{2\ln 2}{\pi}}$$

$$\Delta\nu_0 \, \Delta t_0 = \frac{2\ln 2}{\pi}$$

4. $\Delta\omega_0 = 2\pi\Delta\nu_0$ donc $\Delta\omega_0 \, \Delta t_0 = 2\pi \times \frac{2\ln 2}{\pi} = 4\ln 2.$

exercices

1. $X(\nu) = \int_{-\infty}^{+\infty} e^{-2i\pi\nu t} H(t) e^{-2\pi t} dt = \int_0^{+\infty} e^{-2\pi t} dt \, e^{-2i\pi\nu t}$

$$X(\nu) = \left[\frac{e^{-2\pi t - 2i\pi\nu t}}{-2\pi - 2i\pi\nu} \right]_0^{+\infty} = \frac{1}{2\pi + 2i\pi\nu}$$

$$X(\nu) = \frac{1}{2\pi}\left(\frac{1}{1+i\nu}\right)$$

2. $E = \int_{-\infty}^{+\infty} \left(H(t) e^{-2\pi t}\right)^2 dt = \int_0^{+\infty} e^{-4\pi t} dt$

$$E = \left[\frac{e^{-4\pi t}}{-4\pi}\right]_0^{+\infty} = \frac{1}{4\pi}$$

3. $E = \int_{-\infty}^{+\infty} |X(\nu)|^2 d\nu = \frac{1}{4\pi^2} \times \int_{-\infty}^{+\infty} \frac{d\nu}{1+\nu^2}$

4. Donc $\int_{-\infty}^{+\infty} \frac{d\nu}{1+\nu^2} = \frac{4\pi^2}{4\pi} = \pi$

$\int_{-\infty}^{+\infty} \frac{dt}{1+t^2}$ est la même valeur.

exercice 6

$$\frac{d}{dt}\left(-\frac{\alpha^2}{4}e^{-\frac{2t^2}{\alpha^2}}\right) = -\frac{\alpha^2}{4}\times\left(-\frac{4t}{\alpha^2}\right)e^{-\frac{2t^2}{\alpha^2}}$$

$$= t\, e^{-\frac{2t^2}{\alpha^2}}$$

$$E_{x_\alpha} = \int_{-\infty}^{+\infty}(x_\alpha(t))^2 dt = \int_{-\infty}^{+\infty} e\, t^2 e^{-\frac{t^2}{\alpha^2}\times 2} dt$$

$$= \int_{-\infty}^{+\infty} e\, t \times \frac{d}{dt}\left(-\frac{\alpha^2}{4}e^{-\frac{2t^2}{\alpha^2}}\right) dt$$

$$= \underbrace{\left[e\, t\left(-\frac{\alpha^2}{4}e^{-\frac{2t^2}{\alpha^2}}\right)\right]_{-\infty}^{+\infty}}_{=0}$$

$$-\int_{-\infty}^{+\infty} e\left(-\frac{\alpha^2}{4}e^{-\frac{2t^2}{\alpha^2}}\right) dt$$

$$= \frac{e\alpha^2}{4}\int_{-\infty}^{+\infty} e^{-\frac{2t^2}{\alpha^2}} dt = \frac{e\alpha^2}{4}\times\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\left(\frac{\alpha}{2}\right)^2}} dt$$

$$= \frac{e\alpha^2}{4}\sqrt{2\pi}\times\left(\frac{\alpha}{2}\right) = \frac{e\alpha^3}{8}\sqrt{2\pi}.$$