# Séance 12

## exercice 1

1. $y(t) = 1_{[-1/2, 1/2]}(t - 1/2)$

$$Y(\nu) = \frac{\sin \pi\nu}{\pi\nu} e^{-i\pi\nu} = \frac{1}{2i\pi\nu}\left(e^{i\pi\nu} - e^{-i\pi\nu}\right)e^{-i\pi\nu}$$

$$Y(\nu) = \frac{1 - e^{-2i\pi\nu}}{2i\pi\nu}$$

2. $\frac{d}{d\nu}\left(1 - e^{-2i\pi\nu}\right) = 2i\pi e^{-2i\pi\nu}$

$$\frac{d}{d\nu}\frac{1}{\nu} = -\frac{1}{\nu^2}$$

$$\frac{d}{d\nu} Y(\nu) = \frac{-1}{2i\pi}\left(\frac{1}{\nu^2}\left(1 - e^{-2i\pi\nu}\right) - \frac{2i\pi e^{-2i\pi\nu}}{\nu}\right)$$

3. $x(t) = t\, y(t)$

$$X(\nu) = -\frac{1}{2i\pi}\frac{d}{d\nu} Y(\nu)$$

$$X(\nu) = -\left(\frac{1 - e^{-2i\pi\nu} - 2i\pi\nu\, e^{-2i\pi\nu}}{4\pi^2\nu^2}\right)$$

4.

$$X(\nu) \simeq -\frac{1}{4\pi^2\nu^2}\left(1 - \left(1 - 2i\pi\nu + \frac{(2i\pi\nu)^2}{2}\right) - 2i\pi\nu(1 - 2i\pi\nu)\right)$$

$$X(\nu) \simeq -\frac{1}{4\pi^2\nu^2}\left[2i\pi\nu + 2\pi^2\nu^2 - 2i\pi\nu - 4\pi^2\nu^2\right]$$

$$X(\nu) \simeq \frac{-1}{4\pi^2\nu^2}\left[-2\pi^2\nu^2\right] = \frac{1}{2}$$

Donc $\lim_{\nu \to 0} X(\nu) = \frac{1}{2}$

5. $\int_{-\infty}^{+\infty} x(t)\,dt = \int_0^1 x(t)\,dt = X(0)$

$\int_0^1 x(t)\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$

6. $z(t) = \frac{d}{dt}\left(t\, 1_{[0,1]}(t)\right) = 1_{[0,1]}(t) + t\frac{d}{dt}\left(1_{[0,1]}(t)\right)$

$z(t) = 1_{[0,1]}(t) + t\delta(t) - t\,\delta(t-1)$

$t=1$

$z(t) = 1_{[0,1]}(t) - \delta(t-1)$

7. $Z(\nu) = \frac{\sin \pi\nu}{\pi\nu} e^{-i\pi\nu} - e^{-2i\pi\nu}$

$Z(\nu) = \frac{1 - e^{-2i\pi\nu}}{2i\pi\nu} - e^{-2i\pi\nu}$

$Z(\nu) = \frac{1 - e^{-2i\pi\nu} - 2i\pi\nu e^{-2i\pi\nu}}{2i\pi\nu}$

8. $z(t) = \frac{d}{dt}x(t)$

$Z(\nu) = 2i\pi\nu\, X(\nu)$

9. $X(\nu) = \frac{-1}{4\pi^2\nu^2}\left[1 - e^{-2i\pi\nu} - 2i\pi\nu e^{-2i\pi\nu}\right]$

<u>exercice 2</u>

1. $z(t) = \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\,d\tau$

$z(t) = \int_{-\infty}^{+\infty} x(\tau)\, 1_{[0,1]}(t-\tau)\,d\tau$

$z(t) = \int_{-\infty}^{+\infty} x(\tau)\, 1_{[-1,0]}(\tau-t)\,d\tau$

$z(t) = \int_{t-1}^{t} x(\tau)\,d\tau.$

2. $z(t) = \int_{t-1}^{t} \tau \, d\tau = \left[ \frac{\tau^2}{2} \right]_{t-1}^{t}$

$z(t) = \frac{t^2}{2} - \frac{(t-1)^2}{2}$

$z(t) = \frac{t^2}{2} - \left( \frac{t^2}{2} - t + \frac{1}{2} \right)$

$z(t) = t - \frac{1}{2}$.

3. $TF[x(t)] = TF[t] = \frac{-1}{2i\pi} \frac{d}{d\nu} \left( TF[1] \right)$

$X(\nu) = \frac{i}{2\pi} \frac{d}{d\nu} \delta(\nu) = \frac{i}{2\pi} \delta'(\nu)$

4. $y(t) = 1\!\!1_{[0,1]}(t) = 1\!\!1_{[-1/2, 1/2]}(t - 1/2)$

$Y(\nu) = \frac{\sin \pi\nu}{\pi\nu} e^{-i\pi\nu} = \frac{(e^{i\pi\nu} - e^{-i\pi\nu}) e^{-i\pi\nu}}{2i\pi\nu}$

$Y(\nu) = \frac{1 - e^{-2i\pi\nu}}{2i\pi\nu}$

5. $e^{-2i\pi\nu} \simeq 1 - 2i\pi\nu + \frac{4\pi^2\nu^2(-1)}{2}$

$Y(\nu) \simeq \frac{1}{2i\pi\nu} \left( 1 - 1 + 2i\pi\nu + 2\pi^2\nu^2 \right)$

$Y(\nu) \simeq 1 + \frac{\pi\nu}{i} = 1 - i\pi\nu$

donc $Y(0) = 1$ et $\left. \frac{d}{d\nu} Y(\nu) \right|_{\nu=0} = -i\pi$.

6. $Z(\nu) = X(\nu) Y(\nu) = \frac{i}{2\pi} \delta'(\nu) Y(\nu)$

$Z(\nu) = \frac{i}{2\pi} \left( \delta'(\nu) + i\pi\delta(\nu) \right)$

$Z(\nu) = \frac{i}{2\pi} \delta'(\nu) - \frac{1}{2} \delta(\nu)$.

7. $z(\nu) = TF[t - 1/2] = TF[t] - TF[1/2]$

$$= \frac{i}{2\pi}\delta'(\nu) - \frac{1}{2}\delta(\nu)$$

exercice 3

1. $$vp\left(\frac{1}{t}\right) * 1_{[0,1]}(t) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{-\infty}^{+\infty} \frac{1}{\tau} 1_{[0,1]}(t-\tau)\, 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(\tau)\, d\tau$$

$$1_{[0,1]}(t-\tau) = 1_{[-1,0]}(\tau - t) = 1_{[t-1,t]}(\tau)$$

Si $t < 0$,

$$\int_{-\infty}^{+\infty} \frac{1}{\tau} 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(\tau)\, 1_{[t-1,t]}(\tau)\, d\tau = \int_{t-1}^{t} \frac{d\tau}{\tau} = [\ln(-\tau)]_{t-1}^{t} = \ln\left(\frac{-t}{1-t}\right)$$

si $\varepsilon < -t$

$$= \ln\left|\frac{t}{t-1}\right|$$

Si $t \in ]0,1[$,

$$\int_{-\infty}^{+\infty} \frac{1}{\tau} 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(\tau)\, 1_{[t-1,t]}(\tau)\, d\tau = \int_{t-1}^{-\varepsilon} \frac{d\tau}{\tau} + \int_{\varepsilon}^{t} \frac{d\tau}{\tau}$$

$$= [\ln(-\tau)]_{t-1}^{-\varepsilon} + [\ln \tau]_{\varepsilon}^{t} = \ln\frac{\varepsilon}{1-t} + \ln\left(\frac{t}{\varepsilon}\right)$$

$$= \ln\frac{t}{1-t} = \ln\left|\frac{t}{t-1}\right|$$

Si $t > 1$,

$$\int_{-\infty}^{+\infty} \frac{1}{\tau} 1_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(\tau)\, 1_{[t-1,t]}(\tau)\, d\tau = \int_{t-1}^{t} \frac{d\tau}{\tau} = [\ln \tau]_{t-1}^{t} = \ln\left(\frac{t}{t-1}\right)$$

$$= \ln\left|\frac{t}{t-1}\right|$$

Finalement

$$vp\left(\frac{1}{t}\right) * 1_{[0,1]}(t) = \ln\left|\frac{t}{t-1}\right|$$

$$z(t) = \frac{1}{2} 1_{[0,1]}(t) - \frac{i}{2\pi} \ln\left|\frac{t}{t-1}\right|$$

$Re(z(t)) = \frac{1}{2} \mathbb{1}_{[0,1]}(t)$

$Im(z(t)) = -\frac{1}{2\pi} \ln \left| \frac{t}{t-1} \right|$

Si $t<0$, $Im(z(t)) = -\frac{1}{2\pi} \ln\left(\frac{-t}{1-t}\right) = \frac{1}{2\pi} \ln\left(\frac{1}{1+\frac{1}{-t}}\right)$

$Im(z(t))$ est croissant

$\lim_{t\to-\infty} Im(z(t)) = 0$ $\lim_{t\to 0^-} Im(z(t)) = +\infty$

Si $t \in ]0,1[$, $Im(z(t)) = -\frac{1}{2\pi} \ln\left(\frac{t}{1-t}\right) = -\frac{1}{2\pi} \ln\left(\frac{1}{\frac{1}{t}-1}\right)$

$Im(z(t))$ est décroissant

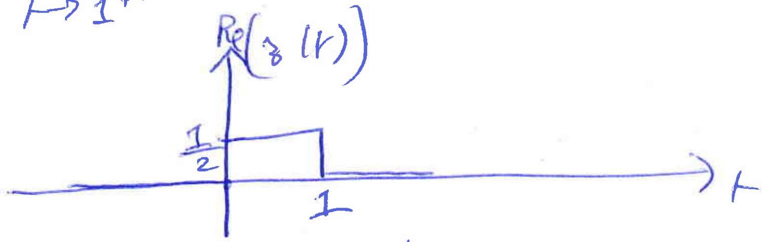
$\lim_{t\to 0^+} Im(z(t)) = +\infty$ $\lim_{t\to 1^-} Im(z(t)) = -\infty$

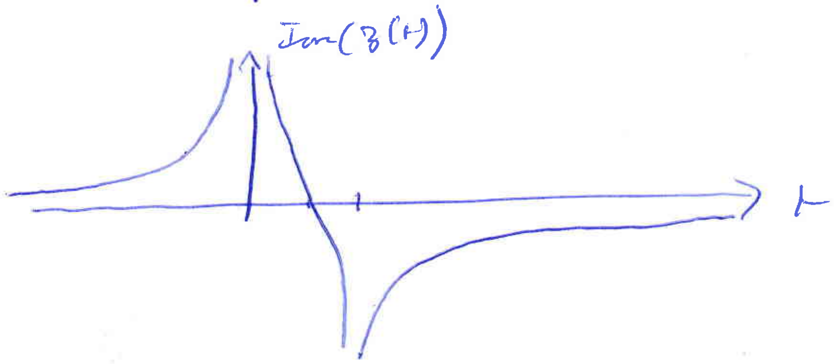
$Im(z(\frac{1}{2})) = 0$

Si $t>1$, $Im(z(t)) = \frac{-1}{2\pi} \ln\left(\frac{t}{t-1}\right) = \frac{-1}{2\pi} \ln\left(\frac{1}{1-\frac{1}{t}}\right)$

$Im(z(t))$ est croissant

$\lim_{t\to 1^+} Im(z(t)) = -\infty$ $\lim_{t\to+\infty} Im(z(t)) = 0$




3. On pose $y_1(t) = y(t) - \frac{1}{2}\delta(t)$

$\frac{d}{d\nu} Y_1(\nu) = \frac{d}{d\nu} TF\left[\frac{1}{2i\pi} vp\frac{1}{t}\right] = TF[-1] = -\delta(\nu)$

$Y_1(\nu) = A - \mathbb{1}_{[0,+\infty[}(\nu)$ avec $A$ inconnu.

Lors que $A = 1/2$, $Y_1(\nu)$ est impair.

Et justement $\frac{1}{2i\pi} vp(\frac{1}{t})$ est impair.

$Y(\nu) = Y_1(\nu) - \frac{1}{2} = -\mathbb{1}_{[0,+\infty[}(\nu)$

4. $Z(\nu) = 0$ si $\nu < 0$

$Z(\nu) = -X(\nu)$ si $\nu > 0$.

pour $\nu > 0$ $|Z(\nu)| = \left| \frac{1 - e^{-2i\pi\nu}}{2i\pi\nu} \right| = \left| e^{-i\pi\nu} \frac{\sin \pi\nu}{\pi\nu} \right| = \left| \frac{\sin \pi\nu}{\pi\nu} \right|$