Séance 10

Ex 10,1

exercice 1

1. $S_{xx}(\nu) = TF[\varphi_{xx}(t)] = TF[x(t) * x(-t)^*]$

$$TF[x(-t)^*] = \int_{-\infty}^{+\infty} x(-t)^* e^{-2i\pi\nu t} dt$$

on effectue le changement de variable $t' = -t$

$$TF[x(-t)^*] = \int_{-\infty}^{+\infty} x(t')^* e^{2i\pi\nu t'} dt'$$

$$TF[x(-t)^*] = TF[x(t)]^* = X(\nu)^*$$

Donc $S_{xx}(\nu) = X(\nu) X(\nu)^* = |X(\nu)|^2$

$$S_{xx}(\nu) = \left(e^{-\pi\nu^2}\right)^2 = e^{-2\pi\nu^2}$$

2. $S_{xx}(\nu) = e^{-\pi(\sqrt{2}\nu)^2} = X(\nu\sqrt{2})$

3. $\varphi_{xx}(t) = TF^{-1}[S_{xx}(\nu)] = TF^{-1}[X(\sqrt{2}\nu)]$

$$= \frac{1}{\sqrt{2}} x\left(\frac{t}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-\pi\left(\frac{t}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{2} e^{-\frac{\pi t^2}{2}}.$$

exercice 2

1. $x(t)$ est causal donc $y(t) = x(t) * x(t)$ est causal.

~~ou encore~~

~~$\to$ Si $t<0$, $y(t) = \int_{-\infty}^{+\infty} x(\tau)\, x(t-\tau) d\tau$~~

et $y(t) = \int_0^t x(\tau)\, x(t-\tau)\, d\tau$ pour $t \geq 0$

donc $y(t) = \left(\int_0^t d\tau\, x(\tau)\, x(t-\tau)\right) \mathbb{1}_{\mathbb{R}_+}(t)$

ou encore.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, x(t-\tau) d\tau = \int_0^{+\infty} x(\tau)\, x(t-\tau)\, d\tau$$

car $x(\tau) = 0$ pour $\tau < 0$.

$$y(t) = \int_0^t x(\tau)\, x(t-\tau)\, d\tau \quad \text{car } x(t-\tau) = 0 \text{ si } \tau > t$$

Si $t < 0$, $y(t) = 0$.

2. $$y(t) = \int_0^t e^{-\alpha\tau}\, e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t} \int_0^t d\tau$$

$$y(t) = t\, e^{-\alpha t}$$

exercice 3

1. $$\varphi_{xx}(-t) = \int_{-\infty}^{+\infty} x(\tau)\, x(\tau - (-t))^{*}\, d\tau$$

changement de variable

$\tau' = \tau + t$, $\tau = \tau' - t$

$$\varphi_{xx}(-t) = \int_{-\infty}^{+\infty} x(\tau' - t)\, x(\tau')^{*}\, d\tau'$$

$$\varphi_{xx}(-t) = \varphi_{xx}(t)^{*}$$

2. Soit $t \geq 0$.

$$\varphi_{xx}(t) = \int_{-\infty}^{+\infty} x(\tau)\, x(\tau - t)^{*}\, d\tau$$

$x(t) = 0$ si $t < 0$, $x(t) \in \mathbb{R}$,

aussi

$$\varphi_{xx}(t) = \int_0^{+\infty} x(\tau)\, x(\tau - t)\, d\tau$$

$x(\tau - t) = 0$ si $\tau < t$

donc

$$\varphi_{xx}(t) = \int_t^{+\infty} x(\tau)\, x(\tau - t)\, d\tau$$

3. Pour $t \geq 0$,

$$\varphi_{xx}(t) = \int_t^{+\infty} e^{-\alpha\tau}\, e^{-\alpha(\tau - t)}\, d\tau$$

$$\varphi_{xx}(t) = e^{\alpha t} \int_t^{+\infty} e^{-2\alpha\tau}\, d\tau = e^{\alpha t} \left[\frac{-1}{2\alpha} e^{-2\alpha\tau} \right]_t^{+\infty}$$

$$\varphi_{xx}(t) = e^{\alpha t}\left(\frac{1}{2\alpha} e^{-2\alpha t} - 0\right)$$

$$\varphi_{xx}(t) = \frac{1}{2\alpha} e^{-\alpha t} = \frac{1}{2\alpha} e^{-\alpha|t|}$$

Pour $t<0$, $\varphi_{xx}(t) = \varphi_{xx}(-t) = \frac{1}{2\alpha} e^{\alpha t} = \frac{1}{2\alpha} e^{-\alpha|t|}$

Donc pour $t\in\mathbb{R}$, $\varphi_{xx}(t) = \frac{1}{2\alpha} e^{-\alpha|t|}$.

exercice 4

1. $E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_0^{+\infty} e^{-2\alpha t} dt$

$$E_x = \left[\frac{-1}{2\alpha} e^{-2\alpha t}\right]_0^{+\infty} = \frac{1}{2\alpha}$$

2. $\varphi_{xx}(0) = \int_{-\infty}^{+\infty} x(\tau)\, x(\tau - 0)^* d\tau$

$$\varphi_{xx}(0) = \int_{-\infty}^{+\infty} |x(\tau)|^2 d\tau = E_x$$

exercice 5

1. $X(\nu) = \int_{-\infty}^{+\infty} e^{-\alpha t}\, 1_{\mathbb{R}_+}(t)\, e^{-2i\pi\nu t} dt$

$$X(\nu) = \left[-\frac{e^{-\alpha t - 2i\pi\nu t}}{\alpha + 2i\pi\nu}\right]_0^{+\infty} = \frac{1}{\alpha + 2i\pi\nu}$$

2. $TF[\varphi_{xx}(t)] = |X(\nu)|^2$

donc $\varphi_{xx}(0) = \int_{-\infty}^{+\infty} |X(\nu)|^2 d\nu$

3. Pour $\alpha = 2\pi$,

$$X(\nu) = \frac{1}{2\pi + 2i\pi\nu} = \frac{1}{2\pi}\left(\frac{1}{1+i\nu}\right)$$

$$|X(\nu)|^2 = \frac{1}{4\pi^2(1+\nu^2)}$$

$$\varphi_{xx}(0) = \frac{1}{2(2\pi)} = \frac{1}{4\pi}$$

Donc $\displaystyle\int_{-\infty}^{+\infty} \frac{d\nu}{1+\nu^2} = 4\pi^2 \int_{-\infty}^{+\infty} \frac{d\nu}{4\pi^2(1+\nu^2)} = 4\pi^2 \int_{-\infty}^{+\infty} |X(\nu)|^2 d\nu$

$$= 4\pi^2 \times \varphi_{xx}(0) = 4\pi^2 \times \frac{1}{4\pi} = \pi$$

exercice 6

$$t e^{-\alpha t} \mathbb{1}_{\mathbb{R}_+}(t) = y(t)$$

$$TF\left[t e^{-\alpha t} \mathbb{1}_{\mathbb{R}_+}(t) \right] = TF[y(t)] = TF[x(t) * x(t)]$$

$$= (X(\nu))^2 = \left(\frac{1}{\alpha + 2i\pi\nu} \right)^2$$