GRAPH THEORY
[5]

Complexity of algorithms – Review (or introduction ?)

Slides adapted from Champion & Chun

Documents are here:

https://www-l2ti.univ-paris13.fr/~viennet/ens/2024-USTH-Graphs

USTH

Université Sorbonne Paris Nord

Questions?

Dictionaries (or maps)

Dictionaries (aka Maps)

Every Programmer's Best Friend

You'll probably use one in almost every programming project.

- Because it's hard to make a big project without needing one sooner or later.

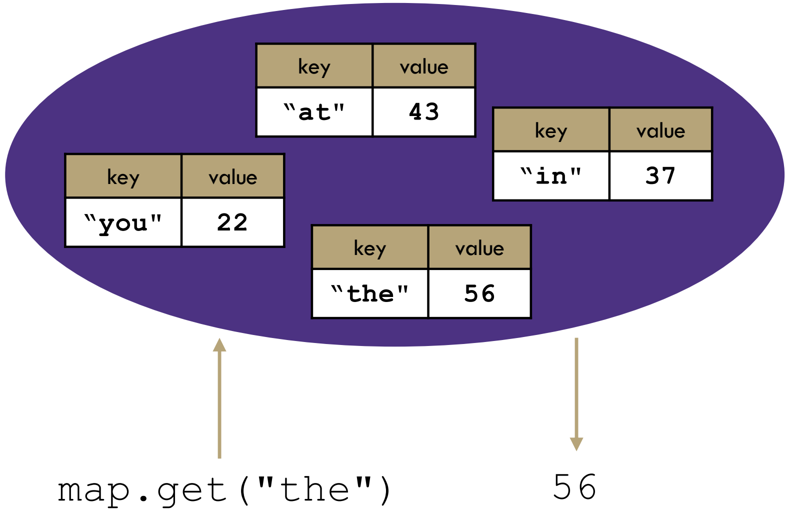
```
// two types of Map implementations
Map<String, Integer> map1 = new HashMap<>();
Map<String, String> map2 =  new TreeMap<>();
```

In Python, builtin type dict :

```
d = {} # empty dictionnary
colors = {
    "red"   : (1, 0, 0),
    "green" : (0, 1, 0),
    "blue"  : (0, 0, 1)
}
```

Review: Maps

map: Holds a set of distinct *keys* and a collection of *values*, where each key is associated with one value.
- a.k.a. "dictionary"



Dictionary ADT

state

Set of items & keys
Count of items

behavior

put(key, item) add item to collection indexed with key
get(key) return item associated with key
containsKey(key) return if key already in use
remove(key) remove item and associated key
size() return count of items

supported operations:

- **put**(*key*, *value*): Adds a given item into collection with associated key,
  - **if the map previously had a mapping for the given key, old value is replaced.**
- **get**(*key*): Retrieves the value mapped to the key
- **containsKey**(key): returns true if key is already associated with value in map, false otherwise
- **remove**(*key*): Removes the given key and its mapped value

Aug → | KEYS | VALUES | → 37.3

KEYS	VALUES
Jan	327.2
Feb	368.2
Mar	197.6
Apr	178.4
May	100.0
Jun	69.9
Jul	32.3
Aug	37.3
Sep	19.0
Oct	37.0
Nov	73.2
Dec	110.9
Annual	1551.0

Implementing a Dictionary with an Array

Dictionary ADT

state
Set of items & keys
Count of items

behavior
put(key, item) add item to collection indexed with key
get(key) return item associated with key
containsKey(key) return if key already in use
remove(key) remove item and associated key
size() return count of items

ArrayDictionary<K, V>

state
Pair<K, V>[] data

behavior
put find key, overwrite value if there. Otherwise create new pair, add to next available spot, grow array if necessary
get scan all pairs looking for given key, return associated item if found
containsKey scan all pairs, return if key is found
remove scan all pairs, replace pair to be removed with last pair in collection
size return count of items in dictionary

```
containsKey('c')
get('d')
put('b', 97)
put('e', 20)
```

0	1	2	3	4
('a', 1)	('b' 97)	('c', 3)	('d', 4)	('e', 20)

Big O Analysis – (if key is the last one looked at / not in the dictionary)

put()	O(N) linear
get()	O(N) linear
containsKey()	O(N) linear
remove()	O(N) linear
size()	O(1) constant

Big O Analysis – (if the key is the first one looked at)

put()	O(1) constant
get()	O(1) constant
containsKey()	O(1) constant
remove()	O(1) constant
size()	O(1) constant

Implementing a Dictionary with Nodes

Dictionary ADT

state

Set of items & keys
Count of items

behavior

put(key, item) add item to collection indexed with key
get(key) return item associated with key
containsKey(key) return if key already in use
remove(key) remove item and associated key
size() return count of items

LinkedDictionary<K, V>

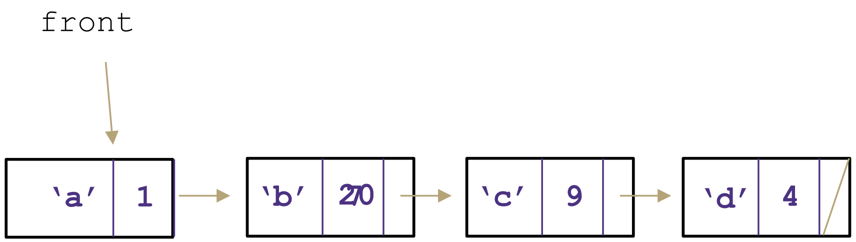
state

```
front
size
```

behavior

```
put if key is unused, create new with
pair, add to front of list, else
replace with new value
get scan all pairs looking for given
key, return associated item if found
containsKey scan all pairs, return if
key is found
remove scan all pairs, skip pair to be
removed
size return count of items in
dictionary
```

```
containsKey('c')
get('d')
put('b', 20)
```



Big O Analysis – (if key is the last one looked at / not in the dictionary)

`put()`	O(N) linear
`get()`	O(N) linear
`containsKey()`	O(N) linear
`remove()`	O(N) linear
`size()`	O(1) constant

Big O Analysis – (if the key is the first one looked at)

`put()`	O(1) constant
`get()`	O(1) constant
`containsKey()`	O(1) constant
`remove()`	O(1) constant
`size()`	O(1) constant

Implementing a Dictionary

Dictionary ADT

state

Set of items & keys
Count of items

behavior

put(key, item) add item to collection indexed with key
get(key) return item associated with key
containsKey(key) return if key already in use
remove(key) remove item and associated key
size() return count of items

Dictionaries are usually implemented using more efficient data structures like **hash tables**

to get O(1) access
(or O(n) in the worst case)

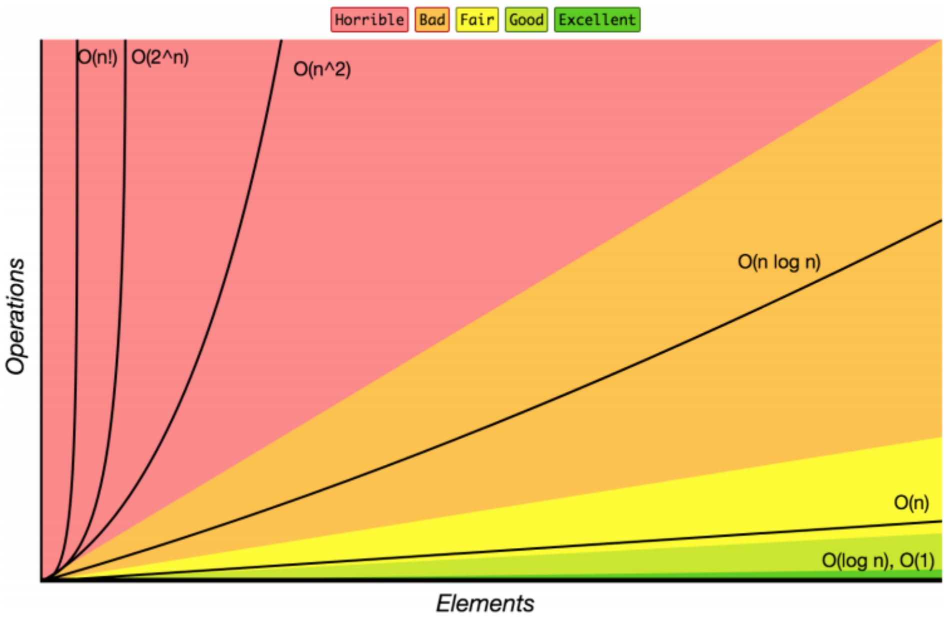
Big O complexity

Review: Complexity Class

Note: You don't have to understand all of this right now – we'll dive into it soon.

complexity class: A category of algorithm efficiency based on the algorithm's relationship to the input size N.

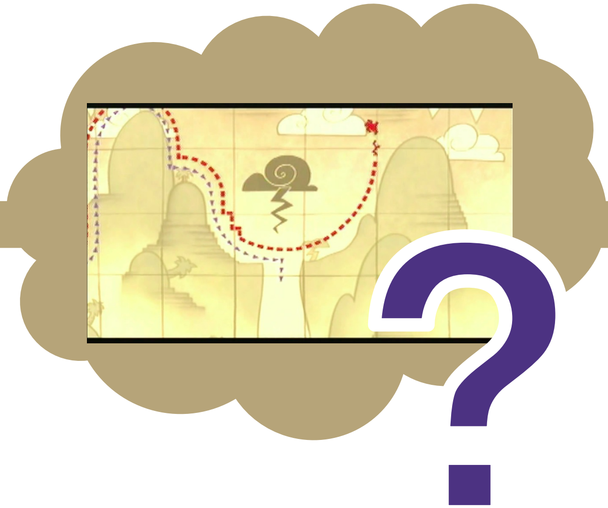
Complexity Class	Big-O	Runtime if you double N	Example Algorithm
constant	O(1)	unchanged	Accessing an index of an array
logarithmic	$O(\log_2 N)$	increases slightly	Binary search
linear	O(N)	doubles	Looping over an array
log-linear	$O(N \log_2 N)$	slightly more than doubles	Merge sort algorithm
quadratic	$O(N^2)$	quadruples	Nested loops!
...	...	...	...
exponential	$O(2^N)$	multiplies drastically	Fibonacci with recursion



Code to Big-Oh

CODE

```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```



BIG-OH

O(n)

General patterns: “O(1) constant is no loops, O(n) is one loop, O(n^2) is nested loops”

But we can go much more in depth: for instance we can explain more about *why*, and how to handle more complex cases when they arise (which they will!)

Meet Algorithmic Analysis


```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```

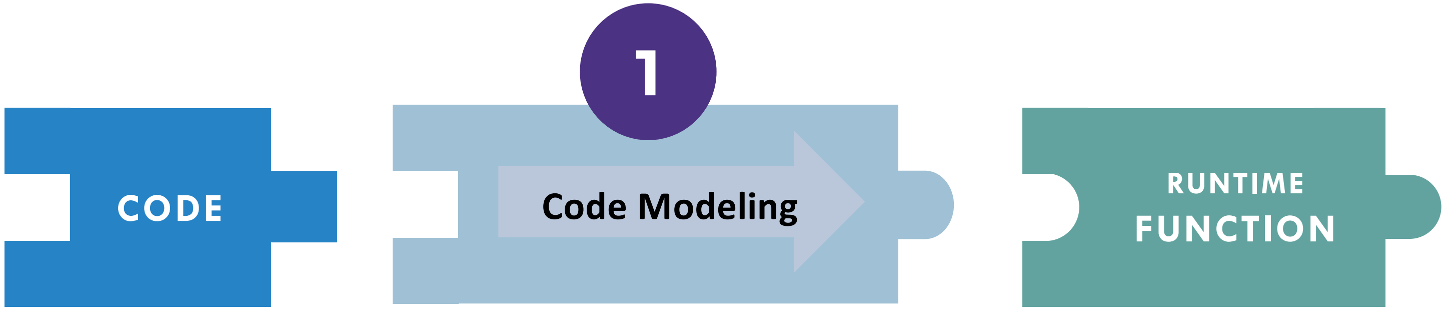
$f(n) = 2n$

$O(n)$

Algorithmic Analysis: The overall process of characterizing code with a complexity class, consisting of:

- Code Modeling: Code → Function describing code's runtime
- Asymptotic Analysis: Function → Complexity class describing asymptotic behavior

Code Modeling



Code Modeling – the process of mathematically representing how many operations a piece of code will run in relation to the input size n.

- Convert from code to a function representing its runtime

What Counts?

We don't know exact runtime of every operation, but for now let's try simplifying assumption: all basic operations take the same time

- Basics:
  - +, -, /, *, %, ==
  - Assignment
  - Returning
  - Variable/array access
- Function Calls
  - Total runtime in body
  - Remember: `new` calls a function (constructor)
- Conditionals
  - Test + time for the followed branch
    - Learn how to reason about branch later
- Loops
  - Number of iterations * total runtime in condition and body

Code Modeling Example 1

```
public void method1(int n) {
    int sum = 0;                   +1
    int i = 0;                     +1
    while (i < n) {                +1
        sum = sum + (i * 3);       +3
        i = i + 1;                 +2
    }
    return sum;                    +1
}
```

Loop runs n times

+6 *n

f(n) = 6n + 3

Code Modeling Example 2

```
public void method2(int n) {
    int sum = 0;                      +1
    int i = 0;                        +1
    while (i < n) {                   +1
        int j = 0;                    +1
        while (j < n) {               +1
            if (j % 2 == 0) {         +2
                // do nothing
            }
            sum = sum + (i * 3) + j;  +4
            j = j + 1;                +2
        }
        i = i + 1;                    +2
    } return sum;                     +1
}
```

This inner loop runs n times

+9 *n

This outer loop runs n times

9n + 4 *n

$f(n) = (9n+4)n + 3$

Exercice

Construct a mathematical function modeling the runtime for the following functions

```
public void mysteryz(ArrayList<String> list) {
    for (int i = 0; i < list.size(); i++) {
        for (int j = 0; j < list.size(); j++) {
      +2 System.out.println(list.get(0));
        }
    }
}
```

n(2)

n(n(2))

Approach

-> start with basic operations, work inside out for control structures

- Each basic operation = +1
- Conditionals = test operations + appropriate branch
- Loop = iterations (loop body)

Where are we?


```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```

$f(n) = 2n$

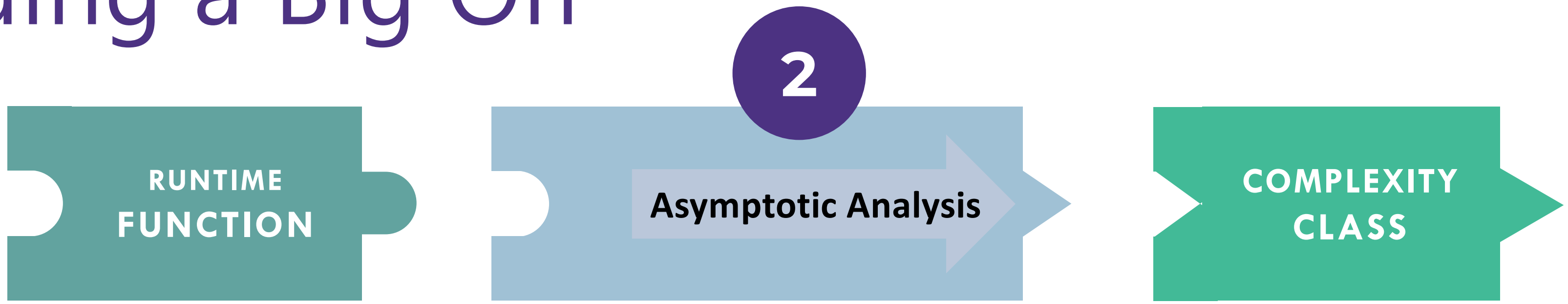
$O(n)$

We just turned a piece of code into a function!

- We'll look at better alternatives for code modeling later

Now to focus on step 2, asymptotic analysis

Finding a Big Oh



We have an expression for $f(n)$. How do we get the $O()$ that we've been talking about?

1. Find the "dominating term" and delete all others.
   - The "dominating" term is the one that is largest as n gets bigger. In this class, often the largest power of n.
2. Remove any constant factors.

$f(n) = (9n+3)n + 3$

$= 9n^2 + 3n + 3$

$\approx 9n^2$

$\approx n^2$

$f(n)$ is $O(n^2)$

Can we really throw away all that info?

Big-Oh is like the "significant digits" of computer science

Asymptotic Analysis: Analysis of function behavior as its input approaches infinity

- We only care about what happens when n approaches infinity
- For small inputs, doesn't really matter: all code is "fast enough"
- Since we're dealing with infinity, constants and lower-order terms don't meaningfully add to the final result. The highest-order term is what drives growth!

Remember our goals:

Simple

We don't care about tiny differences in implementation, want the big picture result

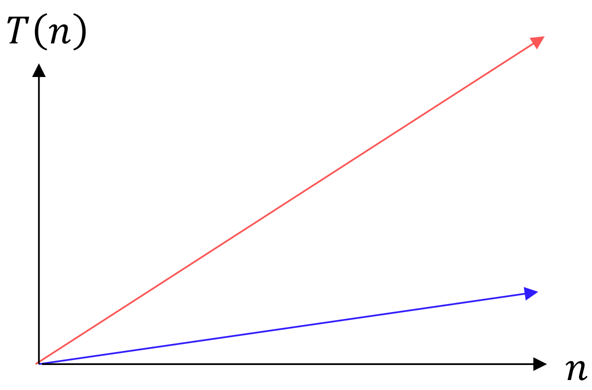
Decisive

Produce a clear comparison indicating which code takes "longer"

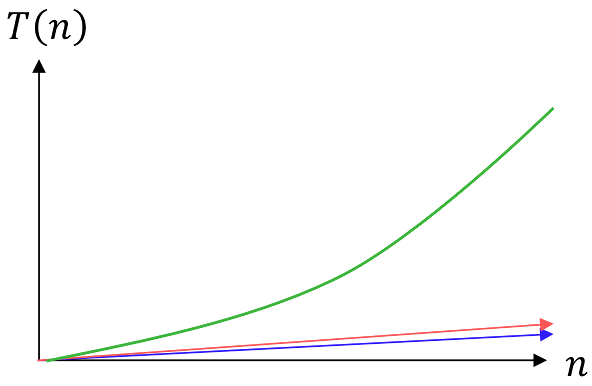
Function growth

Imagine you have three possible algorithms to choose between.
Each has already been reduced to its mathematical model

$f(n) = n$ $g(n) = 4n$ $h(n) = n^2$

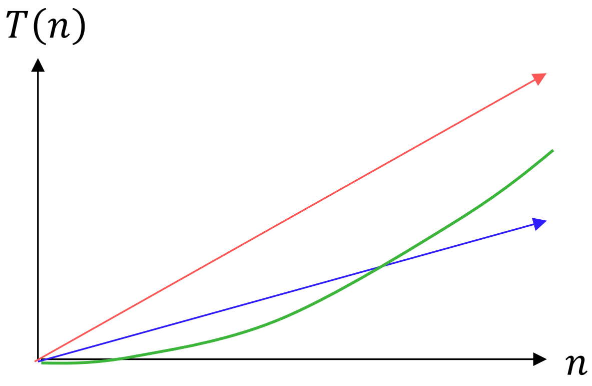


The growth rate for f(n) and g(n) looks very different for small numbers of input



...but since both are linear eventually look similar at large input sizes

whereas h(n) has a distinctly different growth rate



But for very small input values h(n) actually has a slower growth rate than either f(n) or g(n)

Definition: Big-O

We wanted to find an upper bound on our algorithm's running time, but

- We don't want to care about constant factors.
- We only care about what happens as n gets large.

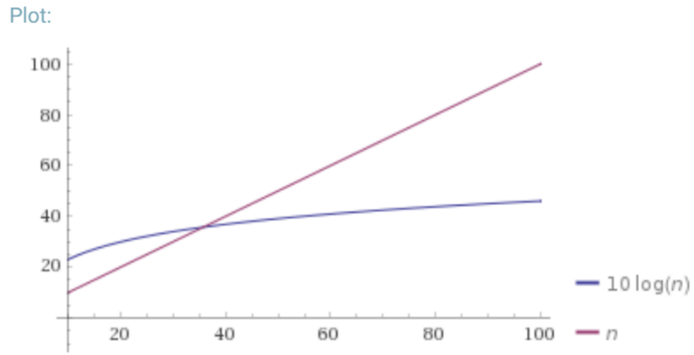
Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

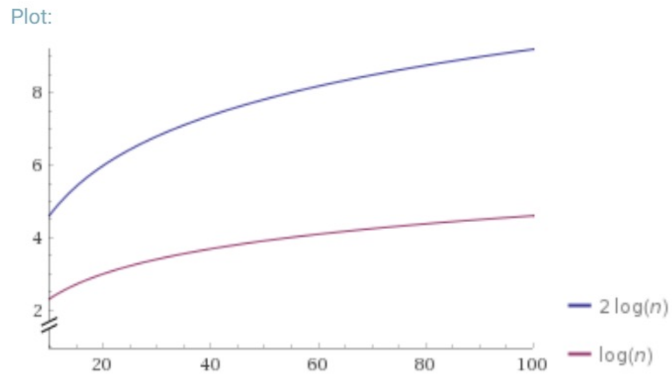
$$f(n) \leq c \cdot g(n)$$

We also say that $g(n)$ "dominates" $f(n)$

Why n_0?



Why c?



Applying Big O Definition

Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Show that $f(n) = 10n + 15$ is $O(n)$

Apply definition term by term

$10n \leq c \cdot n$ when $c = 10$ for all values of n

$15 \leq c \cdot n$ when $c = 15$ for $n \geq 1$

Add up all your truths

$10n + 15 \leq 10n + 15n = 25n$ for $n \geq 1$

Select values for c and n_0 and prove they fit the definition

Take $c = 25$ and $n_0 = 1$

$10n \leq 10n$ for all $values$ of n

$15 \leq 15n$ for $n \geq 1$

So $10n + 15 \leq 25n$ for all $n \geq 1$, as required.

because a c and n_0 exist, $f(n)$ is $O(n)$

Exercise: Proving Big O

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^2 (i.e. that $5n^2 + 3n + 6$ is $O(n^2)$), by finding a c and n_0 that satisfy the definition of domination

$5n^2 + 3n + 6 \le 5n^2 + 3n^2 + 6n^2$ when $n \ge 1$

$5n^2 + 3n^2 + 6n^2 = 14n^2$

$5n^2 + 3n + 6 \le 14n^2$ for $n \ge 1$

$14n^2 \le c*n^2$ for c = ? n >= ?

$\boldsymbol{c} = 14$ & $\boldsymbol{n_0} = 1$

Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \ge n_0$,

$$f(n) \le c \cdot g(n)$$

Note: Big-O definition is just an upper-bound, not always an exact bound

True or False: $10n^2 + 15n$ is $O(n^3)$

It's true – it fits the definition

$10n^2 \leq c \cdot n^3 \; when \; c = 10 \; for \; n \geq 1$

$15n \leq c \cdot n^3 \; when \; c = 15 \; for \; n \geq 1$

$10n^2 + 15n \leq 10n^3 + 15n^3 \leq 25n^3 \; for \; n \geq 1$

$10n^2 + 15n$ is $O(n^3)$ because $10n^2 + 15n \leq 25n^3 \; for \; n \geq 1$

Big-O is just an upper bound that may be loose and not describe the function fully. For example, all of the following are true:

$10n^2 + 15n$ is $O(n^3)$
$10n^2 + 15n$ is $O(n^4)$
$10n^2 + 15n$ is $O(n^5)$
$10n^2 + 15n$ is $O(n^n)$
$10n^2 + 15n$ is $O(n!)$ … and so on

It is (almost always) technically correct to say your code runs in time $O(n!)$
DO NOT TRY TO PULL THIS TRICK IN AN INTERVIEW (or exam).

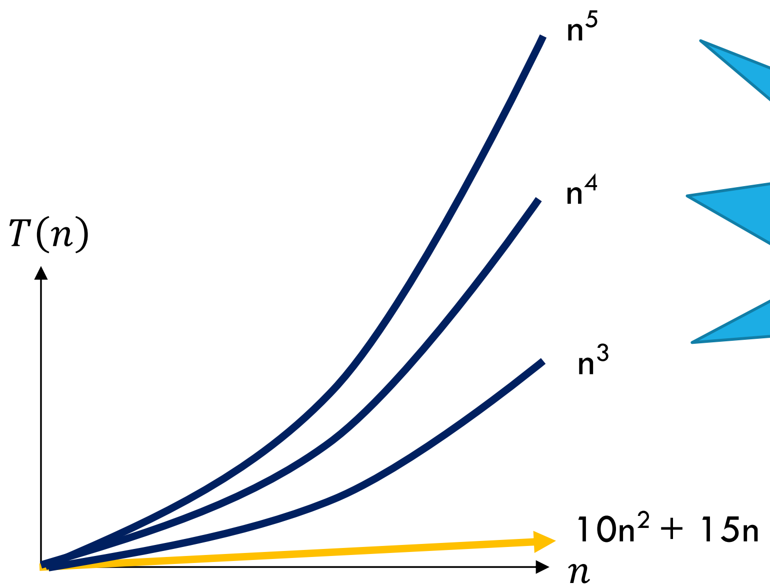
Note: Big-O definition is just an upper-bound, not always an exact bound (plots)

What do we want to look for on a plot to determine if one function is in the big-O of the other?

You can sanity check that your g(n) function (the dominating one) overtakes or is equal to your f(n) function after some point and continues that greater-than-or-equal-to trend towards infinity

$10n^2 + 15n$ is $O(n^3)$
$10n^2 + 15n$ is $O(n^4)$
$10n^2 + 15n$ is $O(n^5)$

… and so on …



The visual representation of big-O and asymptotic analysis is a big idea!

Tight Big-O Definition Plots

If we want the most-informative upper bound, we'll ask you for a simplified, **tight** big-O bound.

$O(n^2)$ is the tight bound for the function f(n) = $10n^2+15n$. See the graph below – the tight big-O bound is the smallest upperbound within the definition of big-O.

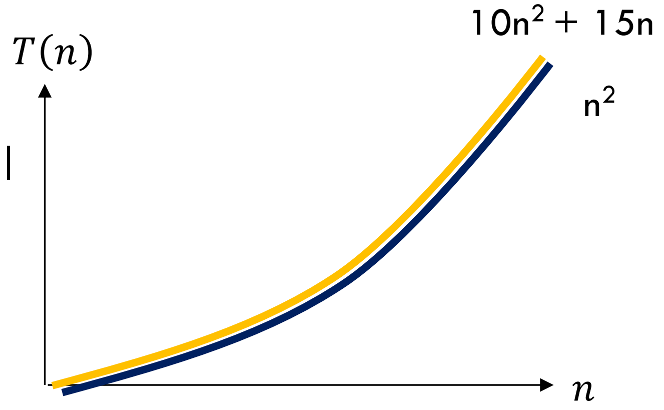
Computer scientists It is almost always technically correct to say your code runs in time $O(n!)$. (Warning: don't try this trick in an interview or exam)

If you zoom out a bunch,

the your tight bound and your function will

be overlapping compared to other

complexity classes.



Questions?

Uncharted Waters: a different type of code model

Find a model $f(n)$ for the running time of this code on input n. What's the Big-O?

```
boolean isPrime(int n){
    int toTest = 2;
    while(toTest < n){
        if(toTest % n == 0) {
            return true;
        } else {
            toTest++;
        }
    }
    return false;
}
```

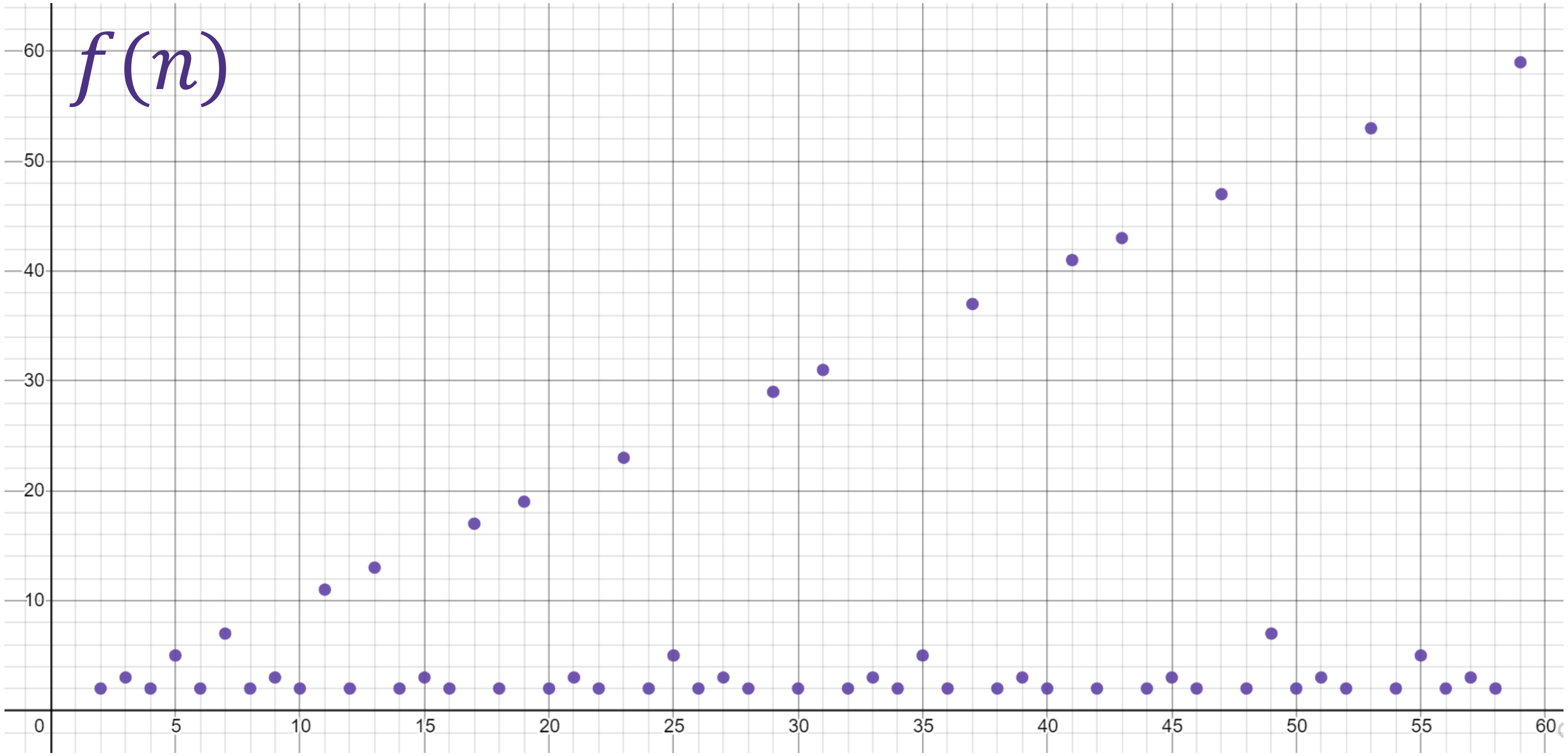
Remember, $f(n)$ = the number of basic operations performed on the input n.

Operations per iteration: let's just call it 1 to keep all the future slides simpler.

Number of iterations?

- Smallest divisor of n

Prime Checking Runtime

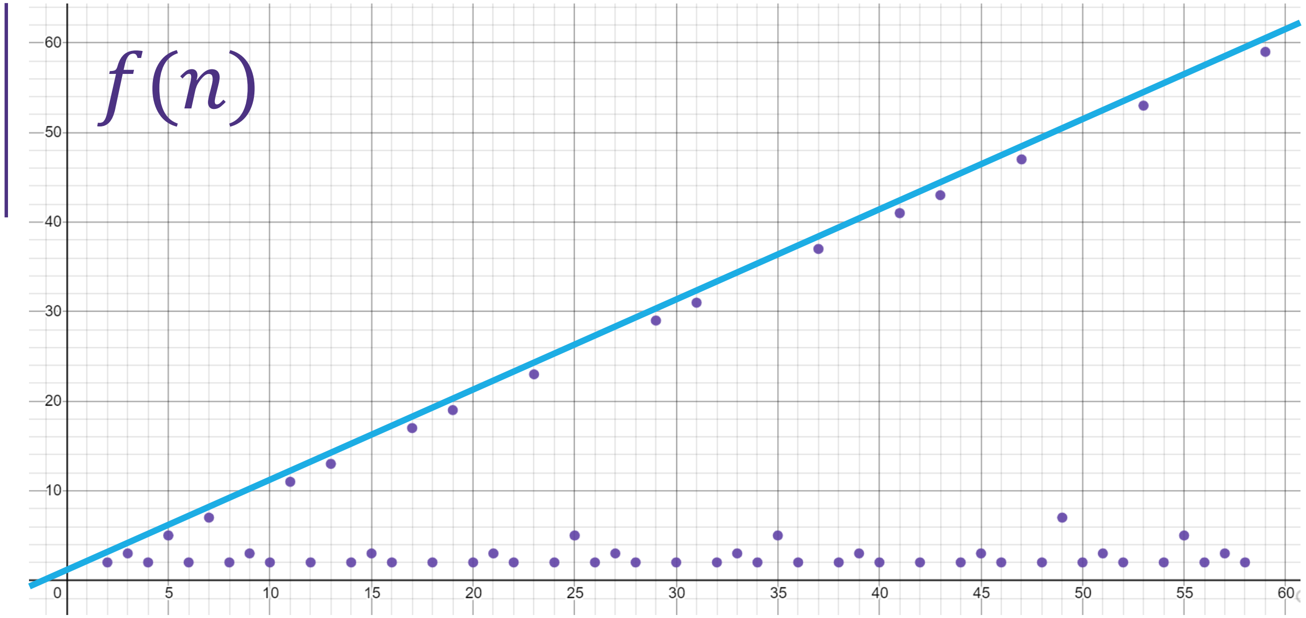


Is the running time of the code $O(1)$ or $O(n)$?

More than half the time we need 3 or fewer iterations. Is it $O(1)$?

But there's still always another number where the code takes n iterations. So $O(n)$?

This is why we have definitions!



Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Is the running time $O(n)$?
Can you find constants c and n_0?

How about $c = 1$ and $n_0 = 5$,
$f(n)$ =smallest divisor of $n \leq 1 \cdot n$ for $n \geq 5$

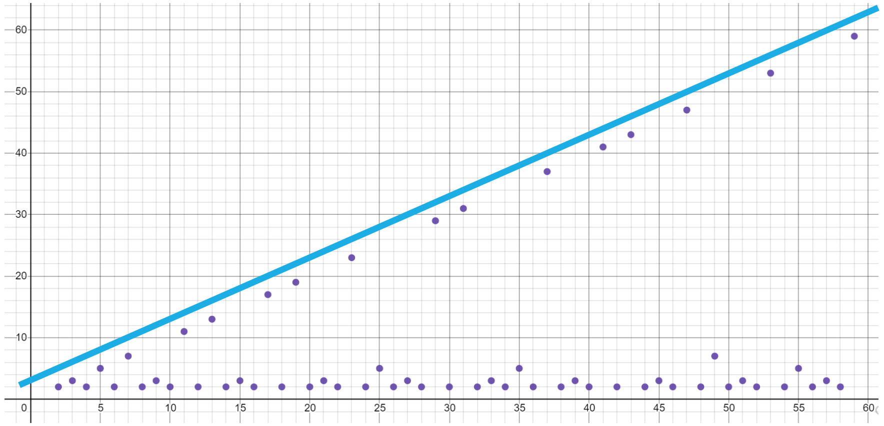
It's $O(n)$ but not $O(1)$

Is the running time $O(1)$?
Can you find constants c and n_0?

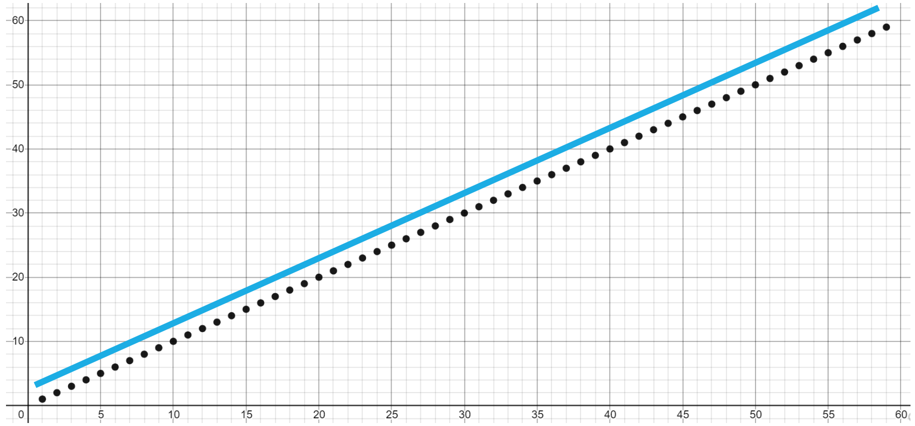
No! Choose your value of c. I can find a prime number k bigger than c.
And $f(k) = k > c \cdot 1$ so the definition isn't met!

Big-O isn't everything

Our prime finding code is $O(n)$. But so is, for example, printing all the elements of a list.



$O(n)$



$O(n)$

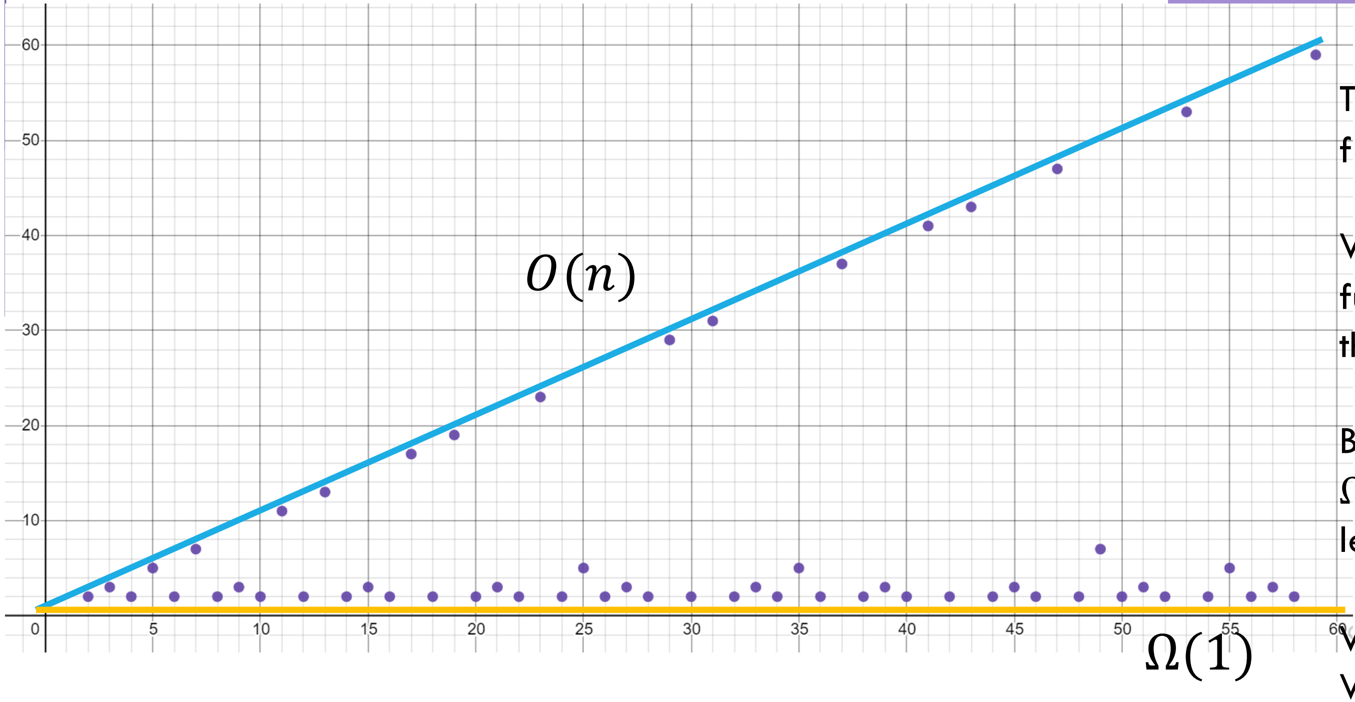
Your experience running these two pieces of code is going to be very different.
It's disappointing that the $O()$ are the same – that's not very precise.
Could we have some way of pointing out the list code always takes AT LEAST n operations?

Big-Ω [Omega]

Big-Omega

$f(n)$ is $\Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \geq c \cdot g(n)$$



The formal definition of Big-Omega is the flipped version of Big-Oh.

When we make Big-Oh statements about a function and say f(n) is O(g(n)) we're saying that f(n) grows at most as fast as g(n).

But with Big-Omega statements like f(n) is Ω(g(n)), we're saying that f(n) will grows at least as fast as g(n).

Visually: what is the lower limit of this function? What is bounded on the bottom by?

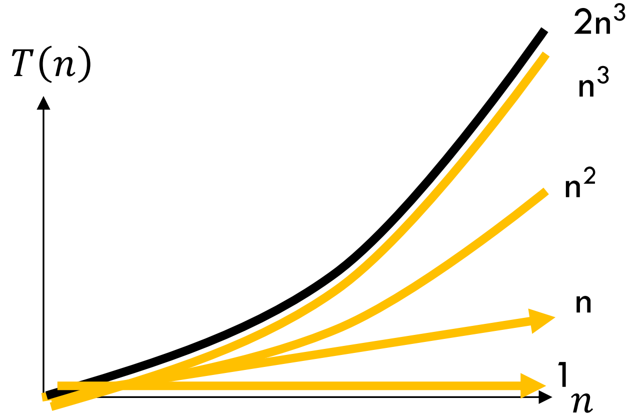
Big-Omega definition Plots

$2n^3$ is $\Omega(1)$

$2n^3$ is $\Omega(\text{n})$

$2n^3$ is $\Omega(n^2)$

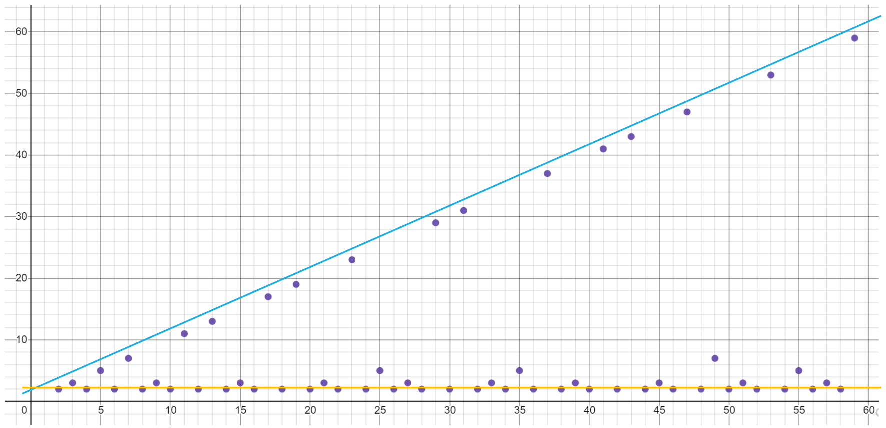
$2n^3$ is $\Omega(n^3)$



$2n^3$ is lowerbounded by all the complexity classes listed above $(1, \text{n}, n^2, n^3)$

Big-O and Big-Ω shown together

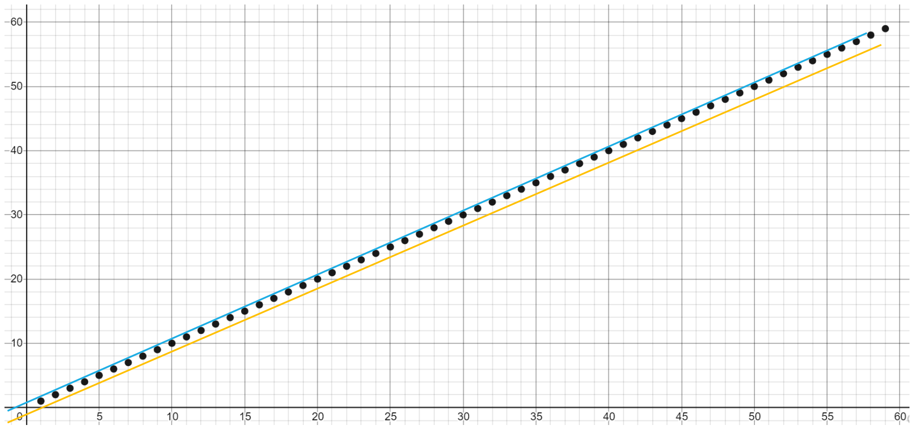
prime runtime function



$O(n)$

$\Omega(1)$

f(n) = n



$O(n)$

$\Omega(\text{n})$

Note: this right graph's tight O bound is O(n) and its tight Omega bound is Omega(n). This is what most of the functions we'll deal with will look like, but there exists some code that would produce runtime functions like on the left.

O, and Omega, and Theta [oh my?]

Big-O is an **upper bound**

- My code takes at most this long to run

Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Big-Omega is a **lower bound**

- **My code takes at least this long to run**

Big-Omega

$f(n)$ is $\Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \geq c \cdot g(n)$$

Big Theta is **"equal to"**

- My code takes "exactly"* this long to run
- *Except for constant factors and lower order terms

Big-Theta

$f(n)$ is $\Theta(g(n))$ if

$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

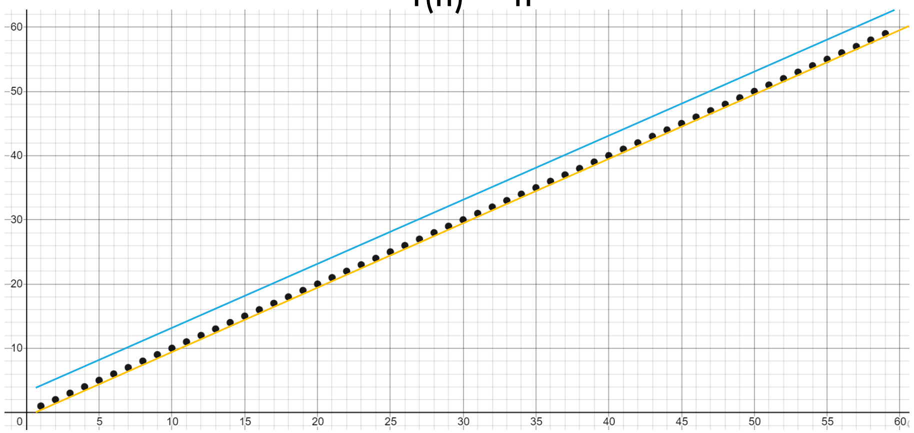
(in other words: there exist positive constants $c1, c2, n_0$ such that for all $n \geq n_0$)

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

O, and Omega, and Theta [oh my?]

Big Theta is **"equal to"**

- My code takes "exactly"* this long to run
- *Except for constant factors and lower order term



$O(n)$ $\Omega(\mathrm{n})$ $\Theta(n)$

Big-Theta

$f(n)$ is $\Theta(g(n))$ if
$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.
(in other words: there exist positive constants $c1, c2, n_0$ such that for all $n \geq n_0$)

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

To define a big-Theta, you expect the tight big-Oh and tight big-Omega bounds to be touching on the graph (meaning they're the same complexity class)

Examples

$4n^2 \in \Omega(1)$

true

$4n^2 \in \Omega(n)$

true

$4n^2 \in \Omega(n^2)$

true

$4n^2 \in \Omega(n^3)$

false

$4n^2 \in \Omega(n^4)$

false

$4n^2 \in O(1)$

false

$4n^2 \in O(n)$

false

$4n^2 \in O(n^2)$

true

$4n^2 \in O(n^3)$

true

$4n^2 \in O(n^4)$

true

Big-O

$f(n) \in O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Big-Omega

$f(n) \in \Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \geq c \cdot g(n)$$

Big-Theta

$f(n) \in \Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.